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2002 J. Phys. A: Math. Gen. 35 2275

(http://iopscience.iop.org/0305-4470/35/9/316)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **35** (2002) 2275–2281 PII: S0305-4470(02)28285-2

On the Mellin transform of a product of two Fox–Wright psi functions

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Received 24 August 2001, in final form 19 November 2001
Published 22 February 2002
Online at stacks.iop.org/JPhysA/35/2275

Abstract

Closed form representations in terms of well known special functions are deduced for the Mellin transform of a product of two Fox–Wright psi functions ${}_1\Psi_1$. An application concerned with the recent analysis of the critical behavior of a variety of distinct physical systems is provided.

PACS number: 02.30.Gp

Mathematics Subject Classification: 33C60, 33C90

1. Introduction

In [1] Diehl and Shpot reconsider the critical behavior of d-dimensional systems with an n-component order parameter at (m, d, n)-Lifschitz points, where a wavevector instability occurs in an m-dimensional subspace of $\mathbb{R}^d$. In their analysis of previously published partly contradictory ϵ-expansion results to the second order in $\epsilon = 4+m/2-d$ $(0 \leqslant \epsilon)$ they introduce the scaling function

$$\Phi(v) = \frac{v^{-(m-2)/2}}{(2\pi)^{d/2}} \int_0^\infty q^{2-\epsilon} J_{\frac{m-2}{2}}(vq) K_{\frac{d-m}{2}-1}(q^2)\, dq \qquad (1.1a)$$

its analogue

$$\Xi(v) = \frac{1}{2} \frac{v^{-(m-2)/2}}{(2\pi)^{d/2}} \int_0^\infty q^{2-\epsilon} J_{\frac{m-2}{2}}(vq) K_{\frac{d-m}{2}-2}(q^2)\, dq \qquad (1.1b)$$

and two improper integrals whose integrands contain powers of these functions, viz,

$$J_{p,\lambda}(m, d) = \int_0^\infty v^{m+p-1} \Phi^\lambda(v)\, dv \qquad (1.2a)$$

and

$$I_1(m, d) = \int_0^\infty v^{m+1} \Phi^2(v) \Xi(v)\, dv \qquad (1.2b)$$

where the integers p, m, λ are such that $p \geqslant 0$, $1 \leqslant m \leqslant d-1$, $\lambda = 2, 3$.

Diehl and Shpot have already noted that the scaling functions $\Phi(v)$ and $\Xi(v)$ defined by equations (1.1) may be expressed essentially as differences whose members are proportional to the generalized hypergeometric function ${}_1F_2[v^4/64]$ (see [1, equations (A2) and (A3)]), i.e. in the form

$$\xi\,{}_1F_2[a; b, c; v^4/64] - \eta v^2\,{}_1F_2[a'; b', c'; v^4/64] \tag{1.3}$$

where ξ, η, and the parameters of each ${}_1F_2$ are functions of m and ϵ. However, if we substitute the appropriate expression (1.3) for the scaling functions into either of equations (1.2), it is evident that we essentially obtain linear combinations of Mellin transforms of products of ${}_1F_2[v^4/64]$ and that each of these transforms diverges.

It shall be the purpose of the present investigation to put equations (1.1) into forms that suggest further insight into the evaluation of the integrals given by equations (1.2). To this end we note that $v^{\frac{1}{2}(m-2)}\Phi(v)$ and $v^{\frac{1}{2}(m-2)}\Xi(v)$ are proportional to specializations of

$$F(v) \equiv \int_0^\infty q^{\alpha-1} J_\mu(vq) K_\omega(q^2)\,\mathrm{d}q \tag{1.4}$$

which we shall evaluate in the next section.

2. Evaluation of $F(v)$

Upon making the transformation $q \mapsto \sqrt{q}$ in equation (1.4) and noting that

$$J_\mu(z) = \frac{(z/2)^\mu}{\Gamma(1+\mu)} \sum_{k=0}^\infty \frac{(-z^2/4)^k}{(1+\mu)_k k!}$$

we obtain

$$F(v) = \frac{1}{2}\,\frac{(v/2)^\mu}{\Gamma(1+\mu)} \sum_{k=0}^\infty \frac{(-v^2/4)^k}{(1+\mu)_k k!} \int_0^\infty q^{\frac{\alpha+\mu}{2}+k-1} K_\omega(q)\,\mathrm{d}q \tag{2.1}$$

where the order of summation and integration have been interchanged. The improper integral in equation (2.1) may be viewed as a Mellin transform of the Macdonald function whose evaluation is well known (see e.g. [4, vol 2, section 2.16.2, equation (2)]). Thus by using the latter result and writing

$$\frac{1}{(1+\mu)_k} = \frac{\sqrt{\pi}}{2^{\mu+k}}\,\frac{\Gamma(1+\mu)}{\Gamma\left(\frac{1+\mu+k}{2}\right)\Gamma\left(\frac{2+\mu+k}{2}\right)}$$

(which is a consequence of the Legendre duplication formula) equation (2.1) gives

$$F(v) = \frac{\sqrt{\pi}}{8} 2^{\frac{1}{2}(\alpha-3\mu)} v^\mu \sum_{k=0}^\infty \frac{\Gamma\left(\frac{\alpha+\mu+2\omega}{4}+\frac{k}{2}\right)\Gamma\left(\frac{\alpha+\mu-2\omega}{4}+\frac{k}{2}\right)}{\Gamma\left(\frac{1+\mu}{2}+\frac{k}{2}\right)\Gamma\left(\frac{2+\mu}{2}+\frac{k}{2}\right)}\,\frac{(-v^2/4)^k}{k!} \tag{2.2}$$

where $\mathrm{Re}\,(\alpha+\mu) > 2|\mathrm{Re}\,\omega|$ for convergence of the integrals in equations (1.4) and (2.1).

However, the series in the latter equation represents a specialization of the Fox–Wright psi function ${}_2\Psi_2[-v^2/4]$ (see [3] for additional references) so that we have finally the following.

Lemma 1. *For* $\mathrm{Re}\,(\alpha+\mu) > 2|\mathrm{Re}\ \omega|$

$$\int_0^\infty q^{\alpha-1} J_\mu(vq) K_\omega(q^2)\,\mathrm{d}q = \frac{\sqrt{\pi}}{8} 2^{\frac{1}{2}(\alpha-3\mu)} v^\mu\,{}_2\Psi_2\left[\begin{matrix}\left(\frac{\alpha+\mu+2\omega}{4}, \frac{1}{2}\right), \left(\frac{\alpha+\mu-2\omega}{4}, \frac{1}{2}\right); \\ \left(\frac{1+\mu}{2}, \frac{1}{2}\right), \left(\frac{2+\mu}{2}, \frac{1}{2}\right);\end{matrix}\ -\frac{v^2}{4}\right].$$

Furthermore, if the series in equation (2.2) is separated or decomposed into even and odd terms in the index k, the latter psi function and integral may be written in terms of two ${}_2F_3[v^4/64]$ *functions (cf [4, vol 2, section 2.16.22, equation (6)]).*

Now recalling that $d = 4 + m/2 - \epsilon$ we may apply lemma 1 to the scaling functions defined by equations (1.1) thus obtaining for $0 \leqslant \epsilon < 2$

$$\Phi(v) = \frac{\sqrt{\pi}2^{-\frac{1}{2}\epsilon-\frac{3}{4}m}}{(2\pi)^{d/2}} {}_1\Psi_1\left[\begin{matrix}\left(1-\frac{\epsilon}{2},\frac{1}{2}\right); \\ \left(\frac{1}{2}+\frac{m}{4},\frac{1}{2}\right);\end{matrix} \; -\frac{v^2}{4}\right] \tag{2.3a}$$

and for $0 \leqslant \epsilon < 1$

$$\Xi(v) = \frac{\sqrt{\pi}2^{-\frac{1}{2}\epsilon-\frac{3}{4}m}}{2(2\pi)^{d/2}} {}_1\Psi_1\left[\begin{matrix}\left(\frac{1}{2}-\frac{\epsilon}{2},\frac{1}{2}\right); \\ \left(\frac{m}{4},\frac{1}{2}\right);\end{matrix} \; -\frac{v^2}{4}\right]. \tag{2.3b}$$

Equation (2.3a) is also derived in [5, section 2].

As intimated above the psi functions in equations (2.3) are readily decomposed thus yielding expressions of the form (1.3). However, the importance of equations (2.3) lies in the fact that the Fox–Wright psi function ${}_p\Psi_q[(\alpha_p,\mu_p);(\beta_q,\nu_q);z]$ (when it converges) is proportional to a specialization of Meijer's G-function provided that the parameters (μ_p) and (ν_q) are positive rational numbers. The latter result is due to Boersma (1962) and may be found in [3, section 5], where concomitant references are cited, and where 2π in each of the three equations of [3, section 5] should be in parentheses. For an introduction to the Fox–Wright psi function and Meijer's G-function, see, for example, [6].

Thus we have immediately from [3, equation (5.1)] the result

$${}_1\Psi_1\left[\begin{matrix}\left(a,\frac{1}{2}\right); \\ \left(b,\frac{1}{2}\right);\end{matrix} \; -\frac{v^2}{4}\right] = \frac{1}{\sqrt{\pi}} G^{2,1}_{1,3}\left(\frac{v^4}{64}\middle| \begin{matrix}1-a \\ 0,\frac{1}{2},1-b\end{matrix}\right) \tag{2.4}$$

which when applied to equations (2.3) gives for $0 \leqslant \epsilon < 2$

$$\Phi(v) = \frac{2^{-\frac{1}{2}\epsilon-\frac{3}{4}m}}{(2\pi)^{d/2}} G^{2,1}_{1,3}\left(\frac{v^4}{64}\middle| \begin{matrix}\frac{\epsilon}{2} \\ 0,\frac{1}{2},\frac{1}{2}-\frac{m}{4}\end{matrix}\right) \tag{2.5a}$$

and for $0 \leqslant \epsilon < 1$

$$\Xi(v) = \frac{2^{-\frac{1}{2}\epsilon-\frac{3}{4}m}}{2(2\pi)^{d/2}} G^{2,1}_{1,3}\left(\frac{v^4}{64}\middle| \begin{matrix}\frac{1}{2}+\frac{\epsilon}{2} \\ 0,\frac{1}{2},1-\frac{m}{4}\end{matrix}\right). \tag{2.5b}$$

Thus, for example, by using equations (2.3) $J_{p,\lambda}(m,d)$ and $I_1(m,d)$ defined by equations (1.2) may be written as specializations proportional to one of the Mellin transforms

$$I(s;x,y,z) \equiv \int_0^\infty v^{s-1} {}_1\Psi_1\left[\begin{matrix}\left(a,\frac{1}{2}\right); \\ \left(b,\frac{1}{2}\right);\end{matrix} \; -\frac{x^2v^2}{4}\right] {}_1\Psi_1\left[\begin{matrix}\left(c,\frac{1}{2}\right); \\ \left(e,\frac{1}{2}\right);\end{matrix} \; -\frac{y^2v^2}{4}\right] \times {}_1\Psi_1\left[\begin{matrix}\left(f,\frac{1}{2}\right); \\ \left(g,\frac{1}{2}\right);\end{matrix} \; -\frac{z^2v^2}{4}\right] \mathrm{d}v \tag{2.6a}$$

$$J(s;x,y) \equiv \int_0^\infty v^{s-1} {}_1\Psi_1\left[\begin{matrix}\left(a,\frac{1}{2}\right); \\ \left(b,\frac{1}{2}\right);\end{matrix} \; -\frac{x^2v^2}{4}\right] {}_1\Psi_1\left[\begin{matrix}\left(c,\frac{1}{2}\right); \\ \left(e,\frac{1}{2}\right);\end{matrix} \; -\frac{y^2v^2}{4}\right] \mathrm{d}v \tag{2.6b}$$

where $x, y, z \neq 0$, and $\mathrm{Re}\,(s) > 0$ for convergence of both integrals at their lower limits.

In the next section we shall evaluate $J(s;x,y) \equiv J(s)$ which is a Mellin transform of a product of two Fox–Wright psi functions ${}_1\Psi_1$. Thus in section 4 we shall be able to obtain a closed form representation for $J_{p,2}(m,d)$ which reduces to the easily computable specializations $J_{0,2}(m,d)$ and $J_{2,2}(m,d)$. Hopefully, the methods used in the analysis of $J(s;x,y)$ will provide some insight into the calculation of $I(s;x,y,z)$ (and therefore $J_{p,3}(m,d)$ and $I_1(m,d)$ also) in terms of known special functions.

3. Representations for $J(s)$ and certain specializations

By using equation (2.4) we see that equation (2.6*b*) may be rewritten as

$$J(s) = \frac{1}{\pi}\int_0^\infty v^{s-1} G_{1,3}^{2,1}\left(\frac{x^4v^4}{64}\,\middle|\,\begin{matrix}1-a\\ 0,\frac{1}{2},1-b\end{matrix}\right) G_{1,3}^{2,1}\left(\frac{y^4v^4}{64}\,\middle|\,\begin{matrix}1-c\\ 0,\frac{1}{2},1-e\end{matrix}\right)\mathrm{d}v.$$

Now making the transformation $v \mapsto 2\sqrt{2}v^{1/4}$ and noting a basic property of the G-function (see e.g. [2, property 2.1]) the latter yields

$$J(s) = \frac{1}{4\pi}\frac{2^{\frac{3}{2}s}}{(xy)^{\frac{s-4}{2}}}\int_0^\infty G_{1,3}^{2,1}\left(vx^4\,\middle|\,\begin{matrix}\frac{1}{2}+\frac{s}{8}-a\\ \frac{s}{8}-\frac{1}{2},\frac{s}{8},\frac{1}{2}+\frac{s}{8}-b\end{matrix}\right) \times G_{1,3}^{2,1}\left(vy^4\,\middle|\,\begin{matrix}\frac{1}{2}+\frac{s}{8}-c\\ \frac{s}{8}-\frac{1}{2},\frac{s}{8},\frac{1}{2}+\frac{s}{8}-e\end{matrix}\right)\mathrm{d}v.$$

The latter integral is evaluated by employing [2, equation (3.10.11)] thus giving after some simplification

$$J(s) = \frac{2^{\frac{3}{2}s}}{4\pi}y^{-s}G_{4,4}^{3,3}\left(\frac{x^4}{y^4}\,\middle|\,\begin{matrix}1-a,1-\frac{s}{4},\frac{1}{2}-\frac{s}{4},e-\frac{s}{4}\\ 0,\frac{1}{2},c-\frac{s}{4},1-b\end{matrix}\right) \tag{3.1}$$

where for convergence of $J(s)$ the parameters a and c are not zero, negative, or half negative integers, $|\arg x| < \pi/4$, $|\arg y| < \pi/4$, and $0 < \mathrm{Re}\,(s) < 4\mathrm{Re}\,(a+c)$.

If we set $c = a$, $e = b$ in equation (3.1), we obtain the following.

Corollary 1. *For* $|\arg x| < \pi/4$, $|\arg y| < \pi/4$, *and* $0 < \mathrm{Re}\,(s) < 8\mathrm{Re}\,(a)$

$$\int_0^\infty v^{s-1}{}_1\Psi_1\left[\begin{matrix}(a,\frac{1}{2});\\ (b,\frac{1}{2});\end{matrix}\; -\frac{x^2v^2}{4}\right]{}_1\Psi_1\left[\begin{matrix}(a,\frac{1}{2});\\ (b,\frac{1}{2});\end{matrix}\; -\frac{y^2v^2}{4}\right]\mathrm{d}v = \frac{2^{\frac{3}{2}s}}{4\pi}y^{-s}G_{4,4}^{3,3}\left(\frac{x^4}{y^4}\,\middle|\,\begin{matrix}1-a,1-\frac{s}{4},\frac{1}{2}-\frac{s}{4},b-\frac{s}{4}\\ 0,\frac{1}{2},a-\frac{s}{4},1-b\end{matrix}\right). \tag{3.2}$$

Assuming arbitrary b note that obviously the parameter a may not be zero, a negative or half negative integer as otherwise the psi functions ${}_1\Psi_1$ *diverge.*

Furthermore, now setting $s = 4b-2$ and $s = 4b$, respectively, in equation (3.2) by exploiting in each case the symmetry of the parameters of the G-function $G_{4,4}^{3,3}(x^4/y^4)$, the latter reduces to a function of lower order (see e.g. [2, p 70] for order reduction properties of the G-function) and we have the results below.

Corollary 2. *For* $|\arg x| < \pi/4$, $|\arg y| < \pi/4$

$$\int_0^\infty v^{4b-3}{}_1\Psi_1\left[\begin{matrix}(a,\frac{1}{2});\\ (b,\frac{1}{2});\end{matrix}\; -\frac{x^2v^2}{4}\right]{}_1\Psi_1\left[\begin{matrix}(a,\frac{1}{2});\\ (b,\frac{1}{2});\end{matrix}\; -\frac{y^2v^2}{4}\right]\mathrm{d}v = \frac{2^{6b}}{32\pi}(y^2)^{1-2b}G_{2,2}^{2,2}\left(\frac{x^4}{y^4}\,\middle|\,\begin{matrix}1-a,\frac{3}{2}-b\\ 0,\frac{1}{2}+a-b\end{matrix}\right) \tag{3.3a}$$

where $\frac{1}{2} < \mathrm{Re}\,(b) < 2\mathrm{Re}\,(a) + \frac{1}{2}$;

$$\int_0^\infty v^{4b-1}{}_1\Psi_1\left[\begin{matrix}(a,\frac{1}{2});\\ (b,\frac{1}{2});\end{matrix}\; -\frac{x^2v^2}{4}\right]{}_1\Psi_1\left[\begin{matrix}(a,\frac{1}{2});\\ (b,\frac{1}{2});\end{matrix}\; -\frac{y^2v^2}{4}\right]\mathrm{d}v = \frac{2^{6b}}{4\pi}(y^2)^{-2b}G_{2,2}^{2,2}\left(\frac{x^4}{y^4}\,\middle|\,\begin{matrix}1-a,\frac{1}{2}-b\\ \frac{1}{2},a-b\end{matrix}\right) \tag{3.3b}$$

where $0 < \mathrm{Re}\,(b) < 2\mathrm{Re}\,(a)$.

Although equations (3.1)–(3.3) and similar type results may be considered elegant, they are by no means immediately useful computationally, since the G-function is just an equivalent notation for a certain contour integral. However, we shall prove lemma 2 below which will enable us to write equations (3.3) in more useful forms.

Lemma 2. *Suppose* $0 < \mathrm{Re}\,(1+\alpha_1-\beta_2) < \mathrm{Re}\,(2+\alpha_1+\alpha_2-\beta_1-\beta_2)$ *and let*

$$\gamma \equiv \frac{\Gamma(1+\alpha_1-\beta_1)\Gamma(1+\alpha_1-\beta_2)\Gamma(1+\alpha_2-\beta_1)\Gamma(1+\alpha_2-\beta_2)}{\Gamma(2+\alpha_1+\alpha_2-\beta_1-\beta_2)}.$$

Then

$$G_{2,2}^{2,2}\left(z\,\middle|\,\begin{matrix}\beta_1,\beta_2\\ \alpha_1,\alpha_2\end{matrix}\right) = \gamma z^{\alpha_1}{}_2F_1\left[\begin{matrix}1+\alpha_1-\beta_2,\,1+\alpha_1-\beta_1;\\ 2+\alpha_1+\alpha_2-\beta_1-\beta_2;\end{matrix}\;1-z\right] \qquad (0<z\leqslant 1) \tag{3.4a}$$

$$= \gamma z^{\beta_2-1}{}_2F_1\left[\begin{matrix}1+\alpha_1-\beta_2,\,1+\alpha_2-\beta_2;\\ 2+\alpha_1+\alpha_2-\beta_1-\beta_2;\end{matrix}\;1-\frac{1}{z}\right] \qquad (z\geqslant 1) \tag{3.4b}$$

and

$$G_{2,2}^{2,2}\left(1\,\middle|\,\begin{matrix}\beta_1,\beta_2\\ \alpha_1,\alpha_2\end{matrix}\right) = \gamma. \tag{3.4c}$$

In order to prove lemma 2 we employ a result of Mathai [2, theorem 2.9, p 109] that represents $G_{p,p}^{p,p}(z)$ in terms of a $(p-1)$-dimensional integral. Specialization of the latter with $p=2$ gives for $0<z<\infty$

$$G_{2,2}^{2,2}\left(z\,\middle|\,\begin{matrix}\beta_1,\beta_2\\ \alpha_1,\alpha_2\end{matrix}\right) = \Gamma(1+\alpha_1-\beta_1)\Gamma(1+\alpha_2-\beta_2)z^{\beta_1-1} \times \int_0^\infty t^{\alpha_1-\beta_2}(1+t)^{\beta_2-\alpha_2-1}(1/z+t)^{\beta_1-\alpha_1-1}\,\mathrm{d}t \tag{3.5}$$

where for convergence of the integral $0<\mathrm{Re}\,(1+\alpha_1-\beta_2)<\mathrm{Re}\,(2+\alpha_1+\alpha_2-\beta_1-\beta_2)$.

The integral in equation (3.5) is readily evaluated for $0<z\leqslant 1$ and for $z\geqslant 1$ by using [4, vol 1, section 2.2.6, equation (24)] in two different obvious ways. Equation (3.4c) follows by setting $z=1$ in either equation (3.4a) or (3.4b) and noting that Gauss's function ${}_2F_1$ reduces to unity when its argument vanishes.

For conciseness we define

$$\gamma_1 \equiv \frac{2^{6b}}{32\pi}\,\frac{\Gamma^2(a)}{\Gamma(2a)}\Gamma\left(b-\frac{1}{2}\right)\Gamma\left(\frac{1}{2}+2a-b\right)$$

and

$$\gamma_2 \equiv \frac{2^{6b}}{4\pi}\,\frac{\Gamma^2(\frac{1}{2}+a)}{\Gamma(1+2a)}\Gamma(1+b)\Gamma(2a-b).$$

Lemma 2 may now be applied to equations (3.3) thus giving

$$\int_0^\infty v^{4b-3}{}_1\Psi_1\left[\begin{matrix}(a,\frac{1}{2});\\ (b,\frac{1}{2});\end{matrix}\;-\frac{x^2v^2}{4}\right]{}_1\Psi_1\left[\begin{matrix}(a,\frac{1}{2});\\ (b,\frac{1}{2});\end{matrix}\;-\frac{y^2v^2}{4}\right]\mathrm{d}v$$

$$= \gamma_1(y^2)^{1-2b}{}_2F_1\left[\begin{matrix}a,b-\frac{1}{2};\\ 2a;\end{matrix}\;1-\frac{x^4}{y^4}\right] \qquad (0<x\leqslant y) \tag{3.6a}$$

$$= \gamma_1(x^2)^{1-2b}{}_2F_1\left[\begin{matrix}a,b-\frac{1}{2};\\ 2a;\end{matrix}\;1-\frac{y^4}{x^4}\right] \qquad (0<y\leqslant x) \tag{3.6b}$$

where $\frac{1}{2} < \mathrm{Re}\,(b) < 2\mathrm{Re}\,(a) + \frac{1}{2}$ and

$$\int_0^\infty v^{4b-1}{}_1\Psi_1\left[\begin{matrix}(a,\frac{1}{2}); \\ (b,\frac{1}{2});\end{matrix}\; -\frac{x^2v^2}{4}\right]{}_1\Psi_1\left[\begin{matrix}(a,\frac{1}{2}); \\ (b,\frac{1}{2});\end{matrix}\; -\frac{y^2v^2}{4}\right]\mathrm{d}v$$

$$= \frac{\gamma_2 x^2}{(y^2)^{1+2b}}\,{}_2F_1\left[\begin{matrix}\frac{1}{2}+a,\,1+b; \\ 1+2a;\end{matrix}\; 1-\frac{x^4}{y^4}\right] \qquad (0 < x \leqslant y) \qquad (3.7a)$$

$$= \frac{\gamma_2 y^2}{(x^2)^{1+2b}}\,{}_2F_1\left[\begin{matrix}\frac{1}{2}+a,\,1+b; \\ 1+2a;\end{matrix}\; 1-\frac{y^4}{x^4}\right] \qquad (0 < y \leqslant x) \qquad (3.7b)$$

where $0 < \mathrm{Re}\,(b) < 2\mathrm{Re}\,(a)$.

Now setting $x = y = 1$ in equations (3.6) and (3.7) we deduce

$$\int_0^\infty v^{4b-3}\left({}_1\Psi_1\left[\begin{matrix}(a,\frac{1}{2}); \\ (b,\frac{1}{2});\end{matrix}\; -\frac{v^2}{4}\right]\right)^2\mathrm{d}v = \frac{2^{6b}}{32\pi}\,\frac{\Gamma^2(a)}{\Gamma(2a)}\Gamma\left(b-\frac{1}{2}\right)\Gamma\left(\frac{1}{2}+2a-b\right) \qquad (3.8a)$$

where $\frac{1}{2} < \mathrm{Re}\,(b) < 2\mathrm{Re}\,(a) + \frac{1}{2}$ and

$$\int_0^\infty v^{4b-1}\left({}_1\Psi_1\left[\begin{matrix}(a,\frac{1}{2}); \\ (b,\frac{1}{2});\end{matrix}\; -\frac{v^2}{4}\right]\right)^2\mathrm{d}v = \frac{2^{6b}}{4\pi}\,\frac{\Gamma^2(\frac{1}{2}+a)}{\Gamma(1+2a)}\Gamma(1+b)\Gamma(2a-b) \qquad (3.8b)$$

where $0 < \mathrm{Re}\,(b) < 2\mathrm{Re}\,(a)$.

4. Representations for $J_{p,2}(m, d)$

We are now ready to apply the results obtained heretofore to $J_{p,\lambda}(m, d)$ for $\lambda = 2$. Combining equations (1.2a) and (2.3a) gives

$$J_{p,2}(m,d) = \frac{\pi 2^{-\epsilon-\frac{3}{2}m}}{(2\pi)^d}\int_0^\infty v^{m+p-1}\left({}_1\Psi_1\left[\begin{matrix}(1-\frac{\epsilon}{2},\frac{1}{2}); \\ (\frac{1}{2}+\frac{m}{4},\frac{1}{2});\end{matrix}\; -\frac{v^2}{4}\right]\right)^2\mathrm{d}v \qquad (4.1)$$

where $d = 4 + m/2 - \epsilon$ and $0 \leqslant \epsilon < 2$. The integral in equation (4.1) may be evaluated in terms of a G-function by setting in equation (3.2) $x = y = 1$, $s = m + p$, $a = 1 - \epsilon/2$, $b = 1/2 + m/4$ thus giving

$$J_{p,2}(m,d) = \frac{2^{-\epsilon+\frac{3}{2}p}}{4(2\pi)^d}G^{3,3}_{4,4}\left(1\,\middle|\,\begin{matrix}\frac{\epsilon}{2},\,1-\frac{m}{4}-\frac{p}{4},\,\frac{1}{2}-\frac{m}{4}-\frac{p}{4},\,\frac{1}{2}-\frac{p}{4} \\ 0,\,\frac{1}{2},\,1-\frac{\epsilon}{2}-\frac{m}{4}-\frac{p}{4},\,\frac{1}{2}-\frac{m}{4}\end{matrix}\right) \qquad (4.2)$$

where $m/4 + p/4 < 2 - \epsilon$. In addition, when $p = 0, 2$ we may immediately apply, respectively, equations (3.8a) and (3.8b) to the integral in equation (4.1) so that

$$J_{0,2}(m,d) = \frac{2^{-2-\epsilon}}{(2\pi)^d}\,\frac{\Gamma^2(1-\frac{\epsilon}{2})}{\Gamma(2-\epsilon)}\Gamma\left(\frac{m}{4}\right)\Gamma\left(2-\frac{m}{4}-\epsilon\right)$$

where $m/4 < 2 - \epsilon$ and

$$J_{2,2}(m,d) = \frac{2^{1-\epsilon}}{(2\pi)^d}\,\frac{\Gamma^2(\frac{3}{2}-\frac{\epsilon}{2})}{\Gamma(3-\epsilon)}\Gamma\left(\frac{3}{2}+\frac{m}{4}\right)\Gamma\left(\frac{3}{2}-\frac{m}{4}-\epsilon\right)$$

where $m/4 < 3/2 - \epsilon$.

The penultimate result for $J_{0,2}(m, d)$ has previously been given by Diehl and Shpot whose derivation for it is intimated in a footnote (see [1, equation (89) and footnote 36]). Moreover, in a private communication Diehl has noted that the generalized quantity $J_{p,2}(m, d)$ given by equation (4.2) could play a role in calculations of the anomalous dimensions of subdominant operators of fourth order in ϕ and second order in $\Delta_\perp$ (see also [1,5]).

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